A Generalization of the Unitary Group*

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1. STATEMENTS

Let $f(t) = f(t_1, \ldots, t_n)$ be a continuous real-valued function defined for all $t \geqslant 0$ (i.e., $t_i \geqslant 0$, $i = 1, \ldots, n$). If A is an $m \times n$ complex matrix and A^* is the conjugate transpose of A, then the nonnegative square roots of the eigenvalues of the $n \times n$ matrix A^*A are called the singular values of A and are denoted here by

$$\alpha_1(A) \geqslant \alpha_2(A) \geqslant \cdots \geqslant \alpha_n(A).$$

Define a real-valued function of the matrix A by

$$\hat{f}(A) = f(\alpha_1(A), \ldots, \alpha_n(A)). \tag{1}$$

This paper is devoted to the following problem: Find all linear transformations T mapping the space $M_{m,n}$ of all $m \times n$ matrices into itself which satisfy

$$\hat{f}(T(A)) = \hat{f}(A) \tag{2}$$

for all $A \in M_{m,n}$. For those f that satisfy the condition that $f(t) = 0$ if

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and only if $t = 0$ we easily verify that the set of all T for which (2) obtains is a group. For clearly it is closed under multiplication, i.e.,

$$\begin{aligned} \hat{f}(T_1T_2(A)) &= \hat{f}(T_1(T_2(A))) \\ &= \hat{f}(T_2(A)) \\ &= \hat{f}(A). \end{aligned}$$

Also A is the zero matrix if and only if every singular value of A is 0. It follows that if $T(A) = 0$ then

$$\begin{aligned} 0 &= \hat{f}(T(A)) \\ &= \hat{f}(A), \end{aligned}$$

and hence that every singular value of A is 0, i.e., A is the zero matrix. Thus any T satisfying (2) must be nonsingular, and

$$\hat{f}(T^{-1}(A)) = \hat{f}(T(T^{-1}(A))) = \hat{f}(A).$$

The proofs of our main results will be made to depend on an earlier result [6] in which the structure of all linear T mapping the set of rank 1 matrices in $M_{m,n}$ into itself is characterized. Thus a second problem we consider is that of characterizing the set of rank 1 matrices in $M_{m,n}$ in terms of the function f. We state our results and relegate the proofs to subsequent sections. The main result of this paper is contained in

THEOREM 1. *Assume that $f(t) = f(t_1, \ldots, t_n)$ is concave, symmetric, strictly increasing in each t_j, $j = 1, \ldots, n$, and $f(0) = 0$. If T is a linear transformation, T: $M_{m,n} \to M_{m,n}$, then T satisfies*

$$\hat{f}(T(A)) = \hat{f}(A), \qquad \textit{all} \quad A \in M_{m,n},$$

if and only if there exist an m-square unitary U and an n-square unitary V such that:

(i) *if $m \neq n$, then*

$$T(A) = UAV, \qquad \textit{all} \quad A \in M_{m,n}; \tag{3}$$

(ii) *if $m = n$, then*

$$T(A) = UAV, \qquad \textit{all} \quad A \in M_{m,m}$$

or

$$T(A) = UA^{\top}V, \qquad \text{all} \quad A \in M_{m,m} \tag{4}$$

($A^{\top}$ is the transpose of A).

The proof of Theorem 1 depends on the following characterization of rank 1 matrices in $M_{m,n}$ which is of independent interest.

THEOREM 2. (i) *If f is strictly increasing in each t_j, $j = 1, \ldots, n$, and $A \in M_{m,n}$ has rank 1, then the only matrix $X \in M_{m,n}$ satisfying both the inequalities*

$$\hat{f}(A \pm X) \leqslant \hat{f}(A) \tag{5}$$

is the zero matrix.

(ii) *Let f be concave, symmetric, and strictly increasing in each t_j, $j = 1, \ldots, n$. If $A \neq 0$ and the only matrix X satisfying both inequalities* (5) *is the zero matrix, then the rank of A is* 1.

Here and subsequently when we write an inequality containing the symbol $\pm$ it is to be understood that the inequality is to hold for both choices of sign.

THEOREM 3. *Let $f(t) = E_r(t_1, \ldots, t_n)$ be the rth elementary symmetric function of $t_1, \ldots, t_n$, $1 \leqslant r \leqslant \min\{m, n\}$. If T: $M_{m,n} \to M_{m,n}$ is a linear transformation, then T satisfies*

$$\hat{f}(T(A)) = \hat{f}(A), \qquad \text{all} \quad A \in M_{m,n},$$

if and only if T has one of the following forms:

(i) *if $r < \min\{m, n\}$, then T satisfies conclusion* (i) *or* (ii) *in Theorem* 1;

(ii) *if $r = m < n$, then T has the form*

$$T(A) = UAV, \qquad \text{all} \quad A \in M_{m,n},$$

where $|\det(U)| = 1$ and V is unitary;

(iii) *if $r = n < m$, then T has the form*

$$T(A) = UAV, \qquad \text{all} \quad A \in M_{m,n},$$

where U is unitary and $|\det(V)| = 1$.

(iv) *if $r = m = n$, then T has the form*

$$T(A) = UAV, \qquad \textit{all} \quad A \in M_{m,m},$$

or

$$T(A) = UA^{\top}V, \qquad \textit{all} \quad A \in M_{m,m},$$

where $|\det(UV)| = 1$.

For $r = 1$ and $m = n$ this result is proved in a forthcoming paper by B. Russo [7]. Russo's result requires the additional assumption that T maps the identity matrix into itself, and his proof depends on a result of one of the present authors [2] in which the structure of those linear T: $M_{n,n} \to M_{n,n}$ which map the unitary group in $M_{n,n}$ into itself is determined. Also in [4] a result similar to Theorem 3 is obtained for linear transformations T holding fixed the elementary symmetric functions of the squares of the singular values.

THEOREM 4. *Assume that $f(t) = 0$ if and only if $t = 0$, and that f is positively homogeneous of degree $p \neq 0$, i.e., $f(\lambda t) = \lambda^p f(t)$, all $\lambda \geqslant 0$, and all $t_j \geqslant 0$, $j = 1, \ldots, n$. If*

$$\hat{f}(T(A)) = \hat{f}(A), \qquad \textit{for all} \quad A \in M_{m,n},$$

then every eigenvalue of T has modulus 1 and every elementary divisor of T is linear.

By specializing f to

$$f(t) = \sum_{j=1}^{n} t_j^{\sigma},$$

where $0 < \sigma \leqslant 1$ in Theorem 1, we have

COROLLARY 1. *If T: $M_{m,n} \to M_{m,n}$ satisfies*

$$\sum_{j=1}^{n} \alpha_j(T(A))^{\sigma} = \sum_{j=1}^{n} \alpha_j(A)^{\sigma}, \qquad \textit{all} \quad A \in M_{m,n},$$

then there exist an m-square unitary U and an n-square unitary V such that:

(i) *if* $m \neq n$, *then*

$$T(A) = UAV, \qquad \textit{all} \quad A \in M_{m,n};$$

(ii) *if* $m = n$, *then*

$$T(A) = UAV, \qquad \textit{all} \quad A \in M_{m,n},$$

or

$$T(A) = UA^{\top}V, \qquad \textit{all} \quad A \in M_{m,n}.$$

In case f is not symmetric, e.g.,

$$f(t) = \sum_{j=1}^{n} c_j t_j, \tag{6}$$

in which $c_j > 0$ but not necessarily equal, we can construct a symmetric function g from f as follows: Define

$$g(t) = \min_{\varphi \in S_n} f(\varphi(t)), \tag{7}$$

where $\varphi(t) = (t_{\varphi(1)}, \ldots, t_{\varphi(n)})$ and S_n is the symmetric group of degree n. For example, if f is given by (6) in which $c_1 \leqslant c_2 \leqslant \cdots \leqslant c_n$, then

$$\hat{g}(A) = \sum_{j=1}^{n} c_j \alpha_j(A).$$

COROLLARY 2. *Let f be concave, strictly increasing in each t_j, $j = 1, \ldots, n$, and satisfy $f(0) = 0$. Define $g(t)$ by* (7). *If T: $M_{m,n} \to M_{m,n}$ is linear and satisfies*

$$\hat{g}(T(A)) = \hat{g}(A), \qquad \textit{all} \quad A \in M_{m,n},$$

then T satisfies conclusion (i) *or* (ii) *in Theorem* 1.

2. PROOFS

We use the following elementary generalization of the standard polar factorization theorem throughout the subsequent arguments. We include a short proof for completeness [1].

LEMMA 1. *Let $A \in M_{m,n}$ and assume the rank of A is k. Then there exist an m-square unitary matrix U and an n-square unitary matrix V such that*

$$UAV = D,$$

where $D \in M_{m,n}$, $d_{ii} = \alpha_i(A)$, $i = 1, \ldots, k$, and $d_{ij} = 0$ otherwise.

Proof. First observe that the $\alpha_i(A)$, $i = 1, \ldots, k$, "fit" in D since $k \leqslant \min\{m, n\}$. The n-square matrix A^*A is positive semidefinite hermitian of rank k with eigenvalues $\alpha_1^2(A), \ldots, \alpha_n^2(A)$. Let $x_1, \ldots, x_n$ be an orthonormal basis of n-tuples which are corresponding eigenvectors of A^*A:

$$A^*Ax_j = \alpha_j^2(A)x_j, \qquad j = 1, \ldots, n.$$

Consider the m-tuples

$$z_j = \frac{Ax_j}{\alpha_j(A)}, \qquad j = 1, \ldots, k. \tag{8}$$

Then, using the standard inner product,

$$\begin{aligned}(z_i, z_j) &= \left(\frac{Ax_i}{\alpha_i(A)}, \frac{Ax_j}{\alpha_j(A)}\right) \\ &= \frac{1}{\alpha_i(A)\alpha_j(A)}(A^*Ax_i, x_j) \\ &= \frac{\alpha_i^2(A)}{\alpha_i(A)\alpha_j(A)}(x_i, x_j) \\ &= \delta_{ij}, \qquad\qquad i, j = 1, \ldots, k.\end{aligned}$$

Moreover, if $j = k + 1, \ldots, n$,

$$Ax_j = 0. \tag{9}$$

For,

$$\begin{aligned}||Ax_j||^2 &= (Ax_j, Ax_j) \\ &= (A^*Ax_j, x_j) \\ &= \alpha_j^2(A)||x_j||^2 \\ &= 0.\end{aligned}$$

Thus the m-tuples $z_1, \ldots, z_k$ are orthonormal and can be augmented to an orthonormal basis $z_1, \ldots, z_k, z_{k+1}, \ldots, z_m$. If we combine (8) and (9) we see that

$$AX = ZD,$$

where X is the n-square matrix whose jth column is x_j, $j = 1, \ldots, n$, Z is the m-square matrix whose jth column is z_j, $j = 1, \ldots, m$, and D is the matrix in the statement of Lemma 1. If we set $U = Z^{-1}$ and $V = X$, then the orthonormality conditions imply immediately that U and V are unitary matrices and that

$$UAV = D.$$

In order to prove Theorem 2 we use an elementary result concerning concave functions of two real variables.

LEMMA 2. *Let h be a concave continuous function defined on the non-negative quadrant and strictly increasing in each variable. If $a = (a_1, a_2)$, $a_1 > 0$ and $a_2 > 0$, then there exists an $x = (x_1, x_2) \neq (0, 0)$ such that $a_1 \pm x_1 > 0$, $a_2 \pm x_2 > 0$ and*

$$h(a \pm x) \leqslant h(a).$$

Proof. Let $r > 0$ be such that every point $P(\theta)$ in the circle $C = \{(a_1 + r\cos\theta, a_2 + r\sin\theta) | 0 \leqslant \theta \leqslant 2\pi\}$ has both coordinates positive. Since h is strictly increasing in each coordinate, $h(P(\theta)) > h(a)$ for $0 < \theta < \pi/2$ and $h(P(\theta)) < h(a)$ for $\pi < \theta < 3\pi/2$. Since h is continuous on C, it follows that there are values θ_1 and θ_2 such that $\pi/2 \leqslant \theta_1 \leqslant \pi$, $3\pi/2 \leqslant \theta_2 \leqslant 2\pi$, and $h(P(\theta_1)) = h(a) = h(P(\theta_2))$. We claim that the line segment L connecting $P(\theta_1)$ and $P(\theta_2)$ does not lie to the left of (a_1, a_2) or, more precisely, there is no point (b_1, b_2) on L with $b_1 < a_1$ and $b_2 < a_2$. For suppose there were a point (b_1, b_2) on L with $b_1 < a_1$ and $b_2 < a_2$. Since $b = (b_1, b_2)$ is on the line segment whose end points are $P(\theta_1)$ and $P(\theta_2)$, there is a σ, $0 < \sigma < 1$, such that $b = \sigma P(\theta_1) + (1 - \sigma)P(\theta_2)$. Thus, using the concavity of h, we obtain

$$\begin{aligned} h(b) &= h(\sigma P(\theta_1) + (1-\sigma)P(\theta_2)) \\ &\geqslant \sigma h(P(\theta_1)) + (1-\sigma)h(P(\theta_2)) \end{aligned}$$

$$= \sigma h(a) + (1 - \sigma) h(a)$$

$$= h(a),$$

which contradicts the fact that, since h is strictly increasing, $h(b) < h(a)$. Thus L does not lie to the left of (a_1, a_2). Consequently, if $P(\theta_3)$ is the end point of the diameter of C which contains $P(\theta_1)$ and (a_1, a_2), then $a_1 + r\cos\theta_3 \leqslant a_1 + r\cos\theta_2$ and $a_2 + r\sin\theta_3 \leqslant a_2 + r\sin\theta_2$. Hence $h(P(\theta_3)) \leqslant h(P(\theta_2)) = h(a)$.

Let $x_1 = r\cos\theta_3$, $x_2 = r\sin\theta_3$, and $x = (x_1, x_2)$. Then $a + x = P(\theta_3)$ and $a - x = P(\theta_1)$, and so $h(a \pm x) \leqslant h(a)$.

Proof of Theorem 2(i). Suppose the rank of A is 1 and suppose $\tilde{X}$ is an $m \times n$ matrix satisfying $\hat{f}(A \pm \tilde{X}) \leqslant \hat{f}(A)$. We wish to show that $\tilde{X} = 0 \in M_{m,n}$.

Invoking Lemma 1, we choose an m-square unitary matrix U and an n-square unitary matrix V such that $UAV = D \in M_{m,n}$, where the only nonzero entry of D is $a_1 = \alpha_1(A)$ in the 1,1 position. Let $X = U\tilde{X}V$. Then $\tilde{X} = 0$ if and only if $X = 0$.

Since $A \pm X$ and $U(A \pm \tilde{X})V$ have the same singular values, it follows that $\hat{f}(A \pm \tilde{X}) = \hat{f}(U(A \pm \tilde{X})V) = \hat{f}(D \pm X) = \hat{f}(X \pm D)$. For convenience we display the $m \times n$ matrix $X \pm D$:

$$X \pm D = \begin{bmatrix} x_{11} \pm a_1 & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}.$$

Now if either m or n is 1, then $X \pm D$ has at most one nonzero singular value, namely

$$(|x_{11} \pm a_1|^2 + |x_{21}|^2 + \cdots + |x_{m1}|^2)^{1/2}$$

or

$$(|x_{11} \pm a_1|^2 + |x_{12}|^2 + \cdots + |x_{1n}|^2)^{1/2}.$$

We proceed to analyze the case $n = 1$. In this case $\hat{f}(A \pm \tilde{X}) \leqslant \hat{f}(A)$ if and only if

$$|x_{11} + a_1|^2 + |x_{21}|^2 + \cdots + |x_{m1}|^2 \leqslant a_1{}^2$$

and

$$|x_{11} - a_1|^2 + |x_{21}|^2 + \cdots + |x_{m1}|^2 \leqslant a_1{}^2,$$

i.e.,

$$|x_{11}|^2 + a_1(x_{11} + \bar{x}_{11}) + a_1{}^2 + |x_{21}|^2 + \cdots + |x_{m1}|^2 \leqslant a_1{}^2$$

and

$$|x_{11}|^2 - a_1(x_{11} + \bar{x}_{11}) + a_1{}^2 + |x_{21}|^2 + \cdots + |x_{m1}|^2 \leqslant a_1{}^2,$$

i.e.,

$$|x_{11}|^2 + \cdots + |x_{m1}|^2 \leqslant \pm\, a_1(x_{11} + \bar{x}_{11}).$$

Hence $|x_{11}|^2 + \cdots + |x_{m1}|^2 = 0$ and so $\tilde{X} = 0$. The analysis of the case $m = 1$ is similar.

Suppose then that both m and n are greater than 1. Let $X(1|1)$ denote the $(m-1) \times (n-1)$ matrix obtained by deleting from X the first row and first column. By Lemma 1 there exist an $(m-1)$-square unitary matrix R and an $(n-1)$-square unitary matrix T such that the $(m-1) \times (n-1)$ matrix $RX(1|1)T = \gamma$. The matrix γ has zeros everywhere except in positions $(1, 1), (2, 2), \ldots, (q, q)$ where the q nonzero singular values of $X(1|1)$ appear. We denote these values by $\gamma_2, \ldots, \gamma_{q+1}$.

We now form the matrix $(1 \dotplus R)(X \pm D)(1 \dotplus T)$ where $\dotplus$ denotes direct sum. Since $1 \dotplus R$ and $1 \dotplus T$ are unitary, it follows that $\hat{f}((1 \dotplus R) \cdot (X \pm D)(1 \dotplus T)) = \hat{f}(X \pm D)$. If $S_\pm$ denotes the matrix $(1 \dotplus R) \cdot (X \pm D)(1 \dotplus T)$, then $S_\pm$ has the block form

$$\begin{bmatrix} x_{11} \pm a_{11} & u \\ v & \gamma \end{bmatrix},$$

where u is $1 \times (n-1)$, v is $(m-1) \times 1$, and γ is the above $(m-1) \times (n-1)$ matrix.

If e is the m-tuple whose only nonzero coordinate is a 1 in the first position, then we have

$$(S_\pm S_\pm^*)_{11} = (S_\pm S_\pm^* e, e) \leqslant \alpha_1{}^2(S_\pm).$$

Thus

$$\sqrt{(S_\pm S_\pm^*)_{11}} \leqslant \alpha_1(S_\pm)$$

and so, since f is strictly increasing in each coordinate,

$$\begin{aligned} f(((S_\pm S_\pm^*)_{11})^{1/2}, 0, \ldots, 0) &\leqslant f(\alpha_1(S_\pm), 0, \ldots, 0) \\ &\leqslant f(\alpha_1(S_\pm), \ldots, \alpha_p(S_\pm), 0, \ldots, 0) \\ &= \hat{f}(S_\pm) = \hat{f}(X \pm D) = \hat{f}(D \pm X) \\ &= \hat{f}(A \pm \tilde{X}) \leqslant \hat{f}(A) \\ &= f(\alpha_1(A), 0, \ldots, 0). \end{aligned}$$

Here p, which is no larger than the smaller of m and n, denotes the rank of $S_\pm$. Since f is strictly increasing in each coordinate, we conclude that

$$(|x_{11} \pm a_1|^2 + uu^*)^{1/2} = ((S_\pm S_\pm^*)_{11})^{1/2} \leqslant a_1. \tag{10}$$

Let $x_{11} = \tau + i\mu$, $i = \sqrt{-1}$. Then

$$\begin{aligned} |x_{11} \pm a_1|^2 &= |\tau + i\mu \pm a_1|^2 \\ &= a_1{}^2 \pm 2a_1\tau + \tau^2 + \mu^2. \end{aligned}$$

Thus there is a choice ε of sign so that $a_1{}^2 + 2\varepsilon a_1\tau + \tau^2 + \mu^2 \geqslant a_1{}^2$ with equality if and only if $\tau = \mu = 0$. But, from (10), $a_1{}^2 + 2\varepsilon a_1\tau + \tau^2 + \mu^2 + uu^* \leqslant a_1{}^2$. Hence $\tau = \mu = uu^* = 0$, i.e., $x_{11} = 0$ and $u = 0$.

Now

$$((S_\pm^* S_\pm)_{11})^{1/2} = (a_1{}^2 + v^*v)^{1/2} \leqslant \alpha_1(S_\pm)$$

and so

$$\begin{aligned} f((a_1{}^2 + v^*v)^{1/2}, 0, \ldots, 0) &\leqslant f(\alpha_1(S_\pm), 0, \ldots, 0) \\ &\leqslant f(\alpha_1(S_\pm), \alpha_2(S_\pm), \ldots, \alpha_p(S_\pm), 0, \ldots, 0) \\ &= \hat{f}(S_\pm) = \hat{f}(A \pm \tilde{X}) \leqslant \hat{f}(A) \\ &= f(a_1, 0, \ldots, 0). \end{aligned}$$

Thus $a_1{}^2 + v^*v \leqslant a_1{}^2$ and so $v = 0$. Hence, letting $b_1 = a_1$, $b_i = \gamma_i$ for $i = 2, \ldots, q + 1$, we have

$$\begin{aligned} \hat{f}(S_\pm) &= f(b_{\varphi(1)}, \ldots, b_{\varphi(q+1)}, 0, \ldots, 0) \\ &\leqslant \hat{f}(A) \\ &= f(b_1, 0, \ldots, 0), \end{aligned}$$

where φ is a permutation of degree $q+1$ for which $b_{\varphi(1)} \geqslant b_{\varphi(2)} \geqslant \cdots \geqslant b_{\varphi(q+1)}$. Now if $\varphi(1) \neq 1$ then $b_{\varphi(1)} \geqslant b_1 > 0$ and so $f(b_{\varphi(1)}, \ldots, b_{\varphi(q+1)}, 0, \ldots, 0) > f(b_1, 0, \ldots, 0)$. Hence $\varphi(1) = 1$. Thus, since

$$f(b_1, b_{\varphi(2)}, \ldots, b_{\varphi(q+1)}, 0, \ldots, 0) \leqslant f(b_1, 0, \ldots, 0)$$

and f is strictly increasing in each coordinate, we have $b_2 = \cdots = b_{q+1} = 0$. Consequently $X = 0$.

Proof of Theorem 2(ii). First, note that, if either $m = 1$ or $n = 1$, then Theorem 2(ii) is trivially true. Suppose that both m and n are greater than 1. If the rank of A is $k > 1$, then $\hat{f}(A) = f(a_1, a_2, \ldots, a_k, 0, \ldots, 0)$, where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k > 0$ and $a_i = \alpha_i(A)$, $i = 1, \ldots, k$. Let h be the function of two variables defined by $h(t_1, t_2) = f(t_1, t_2, a_3, \ldots, a_k, 0, \ldots, 0)$. Then h satisfies the hypothesis of Lemma 2 and so there is an $(x_1, x_2) \neq (0, 0)$ such that $a_1 \pm x_1 > 0$, $a_2 \pm x_2 > 0$, and $h((a_1, a_2) \pm (x_1, x_2)) \leqslant h(a_1, a_2)$. Thus

$$f(a_1 \pm x_1, a_2 \pm x_2, a_3, \ldots, a_k, 0, \ldots, 0) \leqslant f(a_1, a_2, \ldots, a_k, 0, \ldots, 0).$$

Let U be an m-square unitary matrix and V be an n-square unitary matrix such that $UAV = D$ as in Lemma 1. Let $\tilde{X}$ be the $m \times n$ matrix whose only nonzero entries are x_1 in the $(1, 1)$ position and x_2 in the $(2, 2)$ position and let $X = U^* \tilde{X} V^*$. Then we have

$$\begin{aligned} \hat{f}(A \pm X) &= \hat{f}(U(A \pm X)V) \\ &= f(a_1 \pm x_1, a_2 \pm x_2, a_3, \ldots, a_k, 0, \ldots, 0) \\ &\leqslant f(a_1, a_2, \ldots, a_k, 0, \ldots, 0) \\ &= \hat{f}(A). \end{aligned}$$

But $X \neq 0$. Thus we have shown that, if the rank of A is greater than 1, then there is a nonzero matrix X such that $\hat{f}(A \pm X) \leqslant \hat{f}(A)$. From this we conclude that, if A is a nonzero matrix for which $\hat{f}(A \pm X) \leqslant \hat{f}(A)$ implies $X = 0$, then the rank of A is 1. This completes the proof of Theorem 2.

In Section 4 we give an example to show that the assumption of symmetry in part (ii) of Theorem 2 is necessary.

Proof of Theorem 1. Since $f(0, \ldots, 0) = 0$ and f is strictly increasing in each coordinate, it follows that $f(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$. Thus, if $\hat{f}(T(A)) = \hat{f}(A)$ for all $A \in M_{m,n}$, then T is nonsingular.

Suppose now that $A \in M_{m,n}$ has rank 1 and that $Y \in M_{m,n}$ satisfies $\hat{f}(T(A) \pm Y) \leqslant \hat{f}(T(A))$. Since T is nonsingular, we can write Y as $T(X)$ for some $X \in M_{m,n}$ and so we have $\hat{f}(T(A) \pm T(X)) \leqslant \hat{f}(T(A))$. But T is linear, and so we have $\hat{f}(T(A \pm X)) \leqslant \hat{f}(T(A))$, and thus, since (2) holds, $\hat{f}(A \pm X) \leqslant \hat{f}(A)$. But A has rank 1, and so by part (i) of Theorem 2, $X = 0$. Thus $Y = 0$. Hence, if $\hat{f}(T(A) \pm Y) \leqslant \hat{f}(T(A))$, we have $Y = 0$, and so by part (ii) of Theorem 2 we conclude that $T(A)$ has rank 1.

Thus T maps the set of rank 1 matrices in $M_{m,n}$ onto itself. One of the present authors has shown [6] that such a T must have the form:

(i) for $m \neq n$, $T(A) = UAV$ for all A, where U and V are, respectively, $m \times m$ and $n \times n$ nonsingular matrices;

(ii) for $m = n$, $T(A) = UAV$ for all A or $T(A) = UA^{\top}V$ for all A where U and V are $n \times n$ nonsingular matrices.

We proceed to show that U and V are unitary. By Lemma 1, U and V can be written

$$U = U_1 D_1 V_1,$$
$$V = U_2 D_2 V_2,$$

where U_1, V_1 are $m \times m$ unitary matrices, U_2, V_2 are $n \times n$ unitary matrices, and D_1 and D_2 are, respectively, $m \times m$ and $n \times n$ diagonal matrices whose diagonal elements are the singular values of U and V, respectively.

In what follows in this proof we assume that $T(A) = UAV$; however, it should be noted that the arguments we give can be applied to the case $T(A) = UA^{\top}V$ to obtain the same conclusion: that U and V are unitary.

Thus

$$\begin{aligned}
\hat{f}(T(A)) &= \hat{f}(UAV) \\
&= \hat{f}(U_1 D_1 V_1 A U_2 D_2 V_2) \\
&= \hat{f}(D_1 V_1 A U_2 D_2) \\
&= \hat{f}(A) \\
&= \hat{f}(V_1 A U_2)
\end{aligned}$$

for all A. Since V_1 and U_2 are unitary, V_1AU_2 varies over all of $M_{m,n}$ as A does. Hence we have shown that the diagonal matrices D_1 and D_2 have the property that, for all $A \in M_{m,n}$, $\hat{f}(D_1AD_2) = \hat{f}(A)$.

Let E_{ij} denote the $m \times n$ matrix whose only nonzero entry is a 1 in row i and column j. Then E_{ij} has 1 as its only nonzero singular value.

Let $d_{11}, \ldots, d_{1m}$ be the diagonal elements of the diagonal matrix D_1 and let $d_{21}, \ldots, d_{2n}$ be those of D_2. Then the matrix $D_1E_{ij}D_2$ has as its only nonzero entry the positive number $d_{1i}d_{2j}$. Thus the only nonzero singular value of $D_1E_{ij}D_2$ is $d_{1i}d_{2j}$.

Thus

$$\begin{aligned} f(1, 0, \ldots, 0) &= \hat{f}(E_{ij}) \\ &= \hat{f}(D_1E_{ij}D_2) \\ &= f(d_{1i}d_{2j}, 0, \ldots, 0), \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n. \end{aligned}$$

But f is strictly increasing in each coordinate and vanishes only at the zero n-tuple, and so we conclude that $d_{1i}d_{2j} = 1$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. From this it is easily seen that $d_{11} = \cdots = d_{1m} = d_{21}^{-1}$ and $d_{21} = \cdots = d_{2n}$. In other words, $D_1 = d_1I_m$ and $D_2 = d_2I_n$, where I_m and I_n are identity matrices of order m and n, respectively, and $d_1d_2 = 1$.

Thus we have that

$$\begin{aligned} T(A) &= U_1(d_1I_m)V_1AU_2(d_2I_n)V_2 \\ &= d_1d_2U_1V_1AU_2V_2 \\ &= U_1V_1AU_2V_2, \qquad \text{for all} \quad A \in M_{m,n}. \end{aligned}$$

But U_1V_1 is unitary and U_2V_2 is unitary.

This completes the proof of Theorem 1.

We proceed to the *proof of Theorem* 3. First observe, however, that Theorem 3 follows from Theorem 1 when $r = 1$. Thus we assume $r > 1$. Also, Theorem 1 is not applicable to E_r. For, although it is known that $E_r^{1/r}(t_1, \ldots, t_n)$ is concave and symmetric for nonnegative variables [5], it unfortunately is not strictly increasing in each t_j. As a matter of fact, Theorem 2 is not valid for $E_r^{1/r}$, $r > 1$.

LEMMA 3. *If* $T\colon\ M_{m,n} \to M_{m,n}$ *and*

$$\hat{E}_r(T(A)) = \hat{E}_r(A), \qquad \textit{all} \quad A \in M_{m,n}, \tag{11}$$

then T *is nonsingular.*

Proof. Suppose $T(A) = 0$. Then, for any $X \in M_{m,n}$,

$$\begin{aligned}\hat{E}_r(A + X) &= \hat{E}_r(T(A + X))\\ &= \hat{E}_r(T(A) + T(X))\\ &= \hat{E}_r(T(X))\\ &= \hat{E}_r(X).\end{aligned}$$

Let $A = UDV$ as in Lemma 1. Then

$$\hat{E}_r(U^*(A + X)V^*) = \hat{E}_r(D + \tilde{X}),$$

where $\tilde{X} = U^*XV^*$. Let the nonzero elements of D be $d_{11}, \ldots, d_{pp}$, and choose $\tilde{X}$ to be the matrix whose only nonzero entries are 1 in positions $(1, 1), \ldots, (l, l)$, where $l = \min\{m, n\}$. Then

$$\begin{aligned}\hat{E}_r(D + \tilde{X}) &= E_r(d_{11} + 1, \ldots, d_{pp} + 1, 1, \ldots, 1, 0, \ldots, 0)\\ &= \hat{E}_r(\tilde{X})\\ &= E_r(1, \ldots, 1, 0, \ldots, 0).\end{aligned}$$

But, since $r \leqslant \min\{m, n\}$, it follows from the nonnegativity of the d_{ii} that $d_{11} = \cdots = d_{pp} = 0$. Hence $D = 0$ and $A = 0$.

We next define

$$\varphi(A, B, x) = E_r(\alpha_1^2(xA + B), \ldots, \alpha_n^2(xA + B)), \tag{12}$$

where A and B are any matrices in $M_{m,n}$ and $x \geqslant 0$.

LEMMA 4. *If* T *satisfies* (11), *then, for any* A *and* B *in* $M_{m,n}$,

$$\binom{n}{r}\varphi(A, B, x) \geqslant \varphi(T(A), T(B), x), \qquad \textit{all} \quad x \geqslant 0. \tag{13}$$

Proof. We have

$$\begin{aligned}\binom{n}{r}\varphi(A, B, x) &= \binom{n}{r}E_r(\alpha_1^2(xA + B), \ldots, \alpha_n^2(xA + B))\\ &\geqslant [E_r(\alpha_1(xA + B), \ldots, \alpha_n(xA + B))]^2\\ &= (\hat{E}_r(xA + B))^2\\ &= (\hat{E}_r(T(xA + B)))^2\\ &= (\hat{E}_r(xT(A) + T(B)))^2\\ &= [E_r(\alpha_1(xT(A) + T(B)), \ldots, \alpha_n(xT(A) + T(B)))]^2\\ &\geqslant E_r(\alpha_1^2(xT(A) + T(B)), \ldots, \alpha_n^2(xT(A) + T(B)))\\ &= \varphi(T(A), T(B), x).\end{aligned}$$

Both of the inequalities above follow from the nonnegativity of the singular values and the convexity of t^2.

The next result is essentially contained in [4], and we therefore omit the proof.

LEMMA 5. *If $r > 1$, then, for any matrices A and B in $M_{m,n}$, $\varphi(A, B, x)$ is a polynomial in x. Moreover, if $A \neq 0$, then*

$$\deg \varphi(A, B, x) \leqslant 2, \qquad \textit{for all} \quad B \in M_{m,n},$$

if and only if A is of rank 1.

LEMMA 6. *If T satisfies* (11) *and $X \in M_{m,n}$ has rank* 1, *then $T(X)$ has rank* 1.

Proof. If X has rank 1, then, by Lemma 5,

$$\deg \varphi(X, B, x) \leqslant 2, \qquad \text{for all} \quad B \in M_{m,n}.$$

But, in view of (13),

$$\deg \varphi(T(X), T(B), x) \leqslant 2, \qquad \text{for all} \quad B \in M_{m,n}.$$

Hence by Lemma 5 and the nonsingularity of T it follows that $T(X)$ has rank 1.

We can once again invoke the result in [6] which states that T has one of the two forms indicated in (i) or (ii) in the proof of Theorem 1.

To complete the proof of Theorem 3, write $U = U_1DV_1$, $V = U_2GV_2$, where U_1, U_2, V_1, and V_2 are unitary matrices and D and G are diagonal matrices with singular values $d_1, \ldots, d_m$ and $g_1, \ldots, g_n$ of U and V, respectively, on the main diagonal. We consider the case in which T has the form $T(A) = UAV$. Now, if $A \in M_{m,n}$,

$$\begin{aligned}\hat{E}_r(A) &= \hat{E}_r(T(A)) \\ &= \hat{E}_r(UAV) \\ &= \hat{E}_r(U_1DV_1AU_2GV_2) \\ &= \hat{E}_r(D(V_1AU_2)G).\end{aligned}$$

Now

$$\hat{E}_r(V_1AU_2) = \hat{E}_r(A),$$

and as A runs over $M_{m,n}$ so does V_1AU_2. Thus we can conclude that

$$\hat{E}_r(A) = \hat{E}_r(DAG) \tag{14}$$

for all $A \in M_{m,n}$. Let F be an $m \times n$ matrix with 1's in position $(1, 1), \ldots, (r, r)$, 0's elsewhere. If P and Q are, respectively, m-square and n-square permutation matrices, we have

$$\begin{aligned}1 &= \hat{E}_r(F) \\ &= \hat{E}_r(PFQ) \\ &= \hat{E}_r(DPFQG) \\ &= \hat{E}_r(P^{\top}DPFQGQ^{\top}) \\ &= \prod_{i=1}^{r} d_{\sigma(i)}g_{\tau(i)},\end{aligned} \tag{15}$$

where σ is a permutation on $\{1, \ldots, m\}$ corresponding to P and τ is a permutation of $\{1, \ldots, n\}$ corresponding to Q. We first consider part (i) of Theorem 3, in which $r < \min\{m, n\}$. Then (15) immediately implies that $d_1 = \cdots = d_m = d$ and $g_1 = \cdots = g_n = g$ with $dg = 1$. But then

$$\begin{aligned} T(A) &= U_1 d I_m V_1 A U_2 g I_n V_2 \\ &= (U_1 V_1) A (U_2 V_2), \end{aligned}$$

and both U_1V_1 and U_2V_2 are unitary. Conversely, if T has the form indicated in the statement (i), then

$$\hat{E}_r(T(A)) = \hat{E}_r(A).$$

Consider part (ii) of Theorem 3, in which we have $r = m < n$. From (15) we conclude that $g_1 = \cdots = g_n = g$ and

$$g^m \prod_{i=1}^{m} d_i = 1.$$

Then

$$\begin{aligned} T(A) &= U_1 D V_1 A U_2 g I_n V_2 \\ &= g(U_1 D V_1) A (U_2 V_2). \end{aligned}$$

The matrix U_2V_2 is unitary and

$$\begin{aligned} |\det(gU_1DV_1)| &= g^m \det(D) \\ &= g^m \prod_{i=1}^{m} d_i \\ &= 1. \end{aligned}$$

Thus T has the required form. Conversely, suppose

$$T(A) = UAV,$$

where V is n-square unitary and $|\det(U)| = 1$. Then

$$\hat{E}_r(UAV) = \hat{E}_r(UA).$$

The nonzero singular values of the matrix UA are the positive square roots of the nonzero eigenvalues of $(UA)^*(UA)$. But these are precisely the same as the positive square roots of the nonzero eigenvalues of the m-square matrix $(UA)(UA)^*$. But, since $r = m$,

$$\begin{aligned} \hat{E}_r(UA) &= [\det((UA)(UA)^*)]^{1/2} \\ &= |\det(U)| \hat{E}_r(A) \\ &= \hat{E}_r(A). \end{aligned}$$

We omit the entirely analogous verifications of part (iii) and (iv) in Theorem 3.

To prove Theorem 4 let $A \in M_{m,n}$, $A \neq 0$, which satisfies

$$T(A) = \lambda A$$

for some λ. Then

$$\begin{aligned} \hat{f}(A) &= \hat{f}(T(A)) \\ &= \hat{f}(\lambda A) \\ &= |\lambda|^p \hat{f}(A). \end{aligned}$$

Now since $A \neq 0$ it follows that $\hat{f}(A) \neq 0$ and thus $|\lambda| = 1$. Suppose that T has an elementary divisor of degree k, $k \geqslant 2$, corresponding to the eigenvalue λ. Then there exist linearly independent X_1, X_2 in $M_{m,n}$ such that

$$\begin{aligned} T(X_1) &= \lambda X_1, \\ T(X_2) &= \lambda X_2 + X_1. \end{aligned}$$

But then, for any positive integer j,

$$T^j(X_2) = \lambda^j X_2 + j\lambda^{j-1} X_1.$$

Now

$$\begin{aligned} \hat{f}\left(\frac{1}{j}\, T^j(X_2)\right) &= \hat{f}\left(\frac{1}{j}\,[\lambda^j X_2 + j\lambda^{j-1} X_1]\right) \\ &= |\lambda|^{p(j-1)} \hat{f}\left(\frac{1}{j}\, \lambda X_2 + X_1\right) \\ &= \hat{f}\left(\frac{\lambda X_2}{j} + X_1\right). \end{aligned} \tag{16}$$

We also have

$$\hat{f}\left(\frac{1}{j}\, T^j(X_2)\right) = \frac{1}{j^p} \hat{f}(X_2). \tag{17}$$

Taking the limit in (16) and (17), we have

$$\hat{f}(X_1) = 0.$$

But then $X_1 = 0$, a contradiction. Thus T has linear elementary divisors.

3. SOME REMARKS AND QUESTIONS

A linear transformation of the form

$$T(A) = UAV$$

has as a matrix representation the mn-square matrix $U \otimes V^{\top}$, where $\otimes$ indicates Kronecker product [3]. We can easily verify that U and V are uniquely determined by T to within scalar multiples if U and V are nonsingular. For suppose

$$\begin{aligned} T(A) &= U_1AV_1 \\ &= U_2AV_2. \end{aligned}$$

Then

$$U_2^{-1}U_1AV_1V_2^{-1} = A$$

and hence

$$U \otimes V = I_{mn}, \tag{18}$$

where $U = U_2^{-1}U_1$ and $V = (V_1V_2^{-1})^{\top}$. But, from (18),

$$u_{ii}V = I_n, \qquad i = 1, \ldots, m,$$

and

$$u_{ij}V = 0, \qquad i \neq j.$$

Hence V is a multiple of I_n and, similarly, U is a multiple of I_m.

We remark that, for $n = 1$ and $f(t_1) = t_1$,

$$\begin{aligned} \hat{f}(A) &= f(\alpha_1(A)) \\ &= \alpha_1(A) \\ &= \left(\sum_{i=1}^{m} |a_{i1}|^2\right)^{1/2}. \end{aligned}$$

Thus $M_{m,1}$ is just the space of complex m-tuples, $\hat{f}(A)$ is the Euclidean norm of A, and Theorem 1 reduces to the standard result stating that a norm preserving transformation is unitary.

If $\sigma = 2$ in Corollary 1 of Theorem 4, i.e., T: $M_{m,n} \to M_{m,n}$ satisfies

$$\sum_{j=1}^{n} \alpha_j(T(A))^2 = \sum_{j=1}^{n} \alpha_j(A)^2, \qquad \text{all} \quad A \in M_{m,n},$$

it is not necessarily the case T must have the form (i) or (ii). For example, if $T(A)$ is obtained from A by a fixed permutation of the elements of A, then in general there do not exist U and V such that $T(A) = UAV$ or $T(A) = UA^{\mathsf{T}}V$. Two questions arise: For precisely which values of σ is it true that any T: $M_{m,n} \to M_{m,n}$ which satisfies

$$\sum_{j=1}^{n} \alpha_j(T(A))^{\sigma} = \sum_{j=1}^{n} \alpha_j(A)^{\sigma}, \qquad \text{all} \quad A \in M_{m,n}, \tag{19}$$

must have the form (i) or (ii)? If σ is such that T does not necessarily have this form, what is the structure of the group of T satisfying (19)?

We know that, if T satisfies Theorem 4, then T has linear elementary divisors and eigenvalues of modulus 1 and hence is similar to a unitary transformation on $M_{m,n}$ (with respect to the inner product $(A, B) = \operatorname{tr}(B^*A)$ on $M_{m,n}$). We conjecture that T is in fact unitary itself.

A result analogous to Theorem 3 should be available for the completely symmetric functions of the singular values, i.e., we conjecture that, if

$$h_r(t_1, \ldots, t_n) = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant n} t_{i_1}, \ldots, t_{i_r}$$

is the completely symmetric function and

$$\hat{h}_r(T(A)) = \hat{h}_r(A), \qquad \text{all} \quad A \in M_{m,n},$$

then T must have the form (i) or (ii) described in the proof of Theorem 1.

4. AN EXAMPLE

We now give an example to show that in part (ii) of Theorem 2 it is necessary to assume that the function f is symmetric.

Let $m = n = 2$ and let $f(t_1, t_2) = ct_1 + t_2$, where c is real and $c > 1$. Then f is concave and strictly increasing but is not symmetric. Let I denote the 2×2 identity matrix. Then of course the rank of I is 2.

We shall show that the only $X \in M_{2,2}$ satisfying

$$\hat{f}(I \pm X) \leqslant \hat{f}(I)$$

is $X = 0$.

Thus let $X \in M_{2,2}$ and choose a unitary $U \in M_{2,2}$ such that

$$U^*XU = \begin{bmatrix} x_1 & x_3 \\ 0 & x_2 \end{bmatrix} = X_1.$$

Then

$$\hat{f}(I \pm X) = \hat{f}(U^*(I \pm X)U) = \hat{f}(I \pm X_1).$$

It is easily checked that, if $\alpha_1 \geqslant \alpha_2$ are the singular values of $I \pm X_1$ (though the notation does not show it, there are two sets of singular values), then

$$\alpha_1{}^2 + \alpha_2{}^2 = |1 \pm x_1|^2 + |x_3|^2 + |1 \pm x_2|^2,$$

$$\alpha_1 \alpha_2 = |1 \pm x_1| \cdot |1 \pm x_2|.$$

Hence

$$(\alpha_1 + \alpha_2)^2 = (|1 \pm x_1| + |1 \pm x_2|)^2 + |x_3|^2.$$

Therefore

$$\alpha_1 + \alpha_2 = [(|1 \pm x_1| + |1 \pm x_2|)^2 + |x_3|^2]^{1/2} \geqslant |1 \pm x_1| + |1 \pm x_2| \tag{20}$$

with equality if and only if $x_3 = 0$.

Now let $x_1 = a_1 + ib_1$, $x_2 = a_2 + ib_2$, where $i = (-1)^{1/2}$. Then, from (20),

$$\alpha_1 + \alpha_2 \geqslant ((1 \pm a_1)^2 + b_1{}^2)^{1/2} + ((1 \pm a_2)^2 + b_2{}^2)^{1/2} \geqslant |1 \pm a_1| + |1 \pm a_2|, \tag{21}$$

$$\alpha_1 + \alpha_2 \geqslant |2 \pm (a_1 + a_2)| \tag{22}$$

with equality holding in (21) if and only if $b_1 = b_2 = 0$. Clearly, then, there is a choice ε of sign in (22) such that $\alpha_1 + \alpha_2 \geqslant 2$. For this choice of sign, $\alpha_1 \geqslant 1$.

Let $c = 1 + \delta$ with $\delta > 0$. Then

$$\begin{aligned} \hat{f}(I + \varepsilon X) &= c\alpha_1 + \alpha_2 \\ &= (\alpha_1 + \alpha_2) + \delta\alpha_1 \\ &\geqslant 2 + \delta. \end{aligned} \tag{23}$$

But if $\hat{f}(I \pm X) \leqslant \hat{f}(I)$ we must have

$$\begin{aligned} c\alpha_1 + \alpha_2 &\leqslant c + 1 \\ &= 2 + \delta. \end{aligned}$$

Hence equality holds in (23) and we have

$$\alpha_1 + \alpha_2 = 2$$

and

$$\alpha_1 = \alpha_2 = 1.$$

Thus $b_1 = b_2 = x_3 = 0$. Therefore

$$1 = \alpha_1 = |1 + \varepsilon a_1|$$

or

$$1 = \alpha_1 = |1 + \varepsilon a_2|$$

and

$$1 = \alpha_2 = |1 + \varepsilon a_1|$$

or

$$1 = \alpha_2 = |1 + \varepsilon a_2|.$$

Thus $a_1 = a_2 = 0$ and so $X = 0$.

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